

*Mathematics and
Reality*

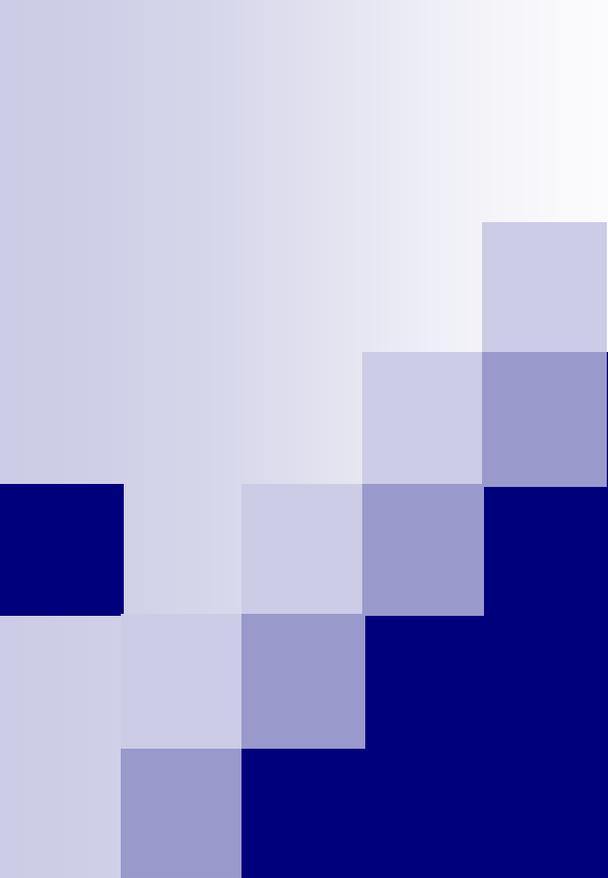
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*Singapore Mathematical Society Lecture
7 March 2008*



Mathematics attempts to...

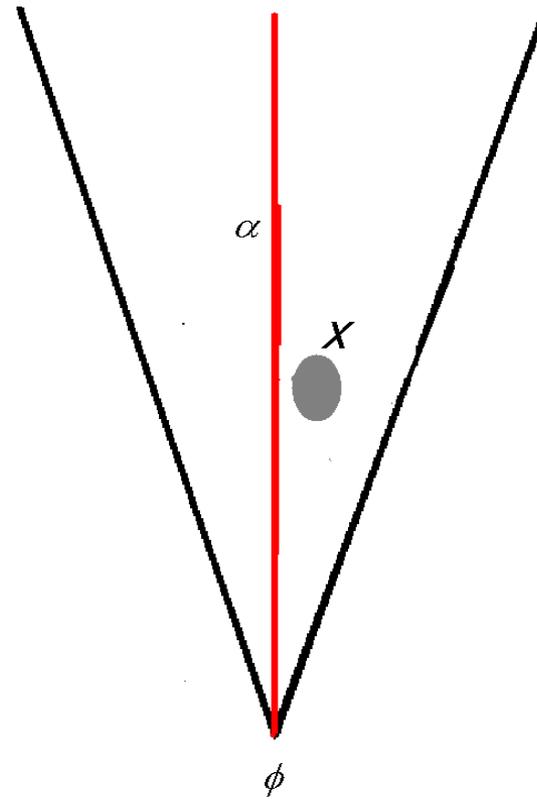
- Understand reality
- Interpret reality
- Model reality
- Simulate reality
- Predict reality



Does Mathematics
Have Its *Own Reality*?

The Universe \mathcal{V} of Sets

- Begin with the empty set ϕ (at the root)
- If X is in \mathcal{V} , so is its power set.
- The “union” of sets already in \mathcal{V} is in \mathcal{V} .
- Practically all of mathematics may be done in \mathcal{V}



Do we understand \mathcal{V} ?

- Kurt Gödel (1906—1976):
- There is a universe \mathcal{L} of sets that is *contained in every* other universe.



Gödel and Einstein



Mom, Kurt, Dad and Rudolph

The Constructible Universe \mathcal{L}

- Every set in \mathcal{L} is “definable” in a very precise sense.
- This places a strong restriction on what may be admitted as sets in \mathcal{L} . For example, why must every set be “definable”?

\mathcal{V} and \mathcal{L}

- Gödel: No contradiction to assume \mathcal{L} IS the *real universe of sets* ($\mathcal{V} = \mathcal{L}$). In fact, in \mathcal{L} :
- Axiom of Choice is true.
- (Hilbert's First Problem) Continuum Hypothesis is true.

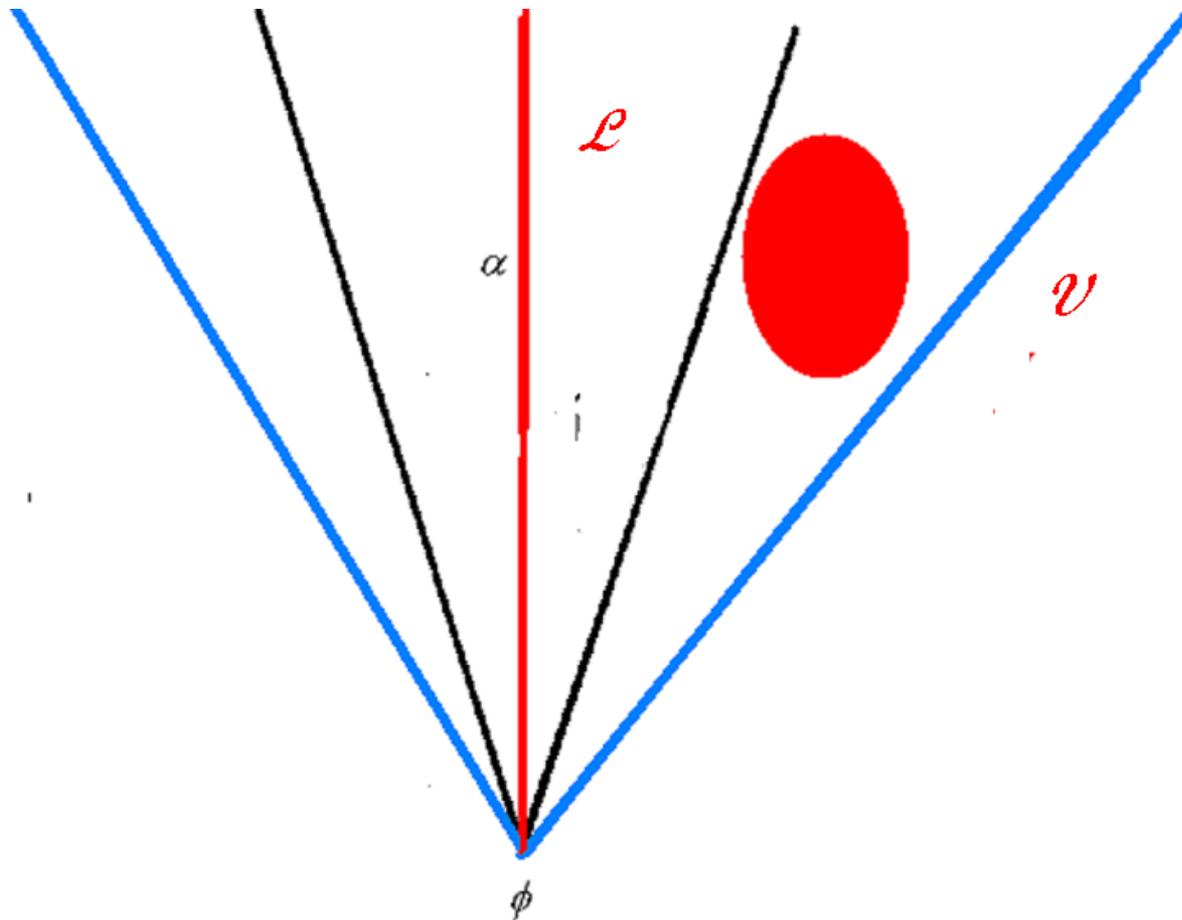
But is $\mathcal{V} = \mathcal{L}$ true?

\mathcal{V} and \mathcal{L}

- Paul Cohen (1934—2007): No contradiction to assume \mathcal{L} is *NOT* the real universe of sets. In fact,
- There is a \mathcal{V} in which Axiom of Choice is false
- There is a \mathcal{V} in which Continuum Hypothesis is false.

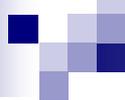


\mathcal{V} and \mathcal{L}



The Basic Question

- Is there a universe of mathematical reality?
- (Surely?) AC and CH are either true or false in the real mathematical universe.
- (Surely?) there can be only *ONE reality*.
- But can we tell which?



Gödel and Cohen's Theorem

If (the set of axioms of) mathematics is consistent, then we will never know which is the real universe of sets.

—Ultimate limit of mathematical knowledge?

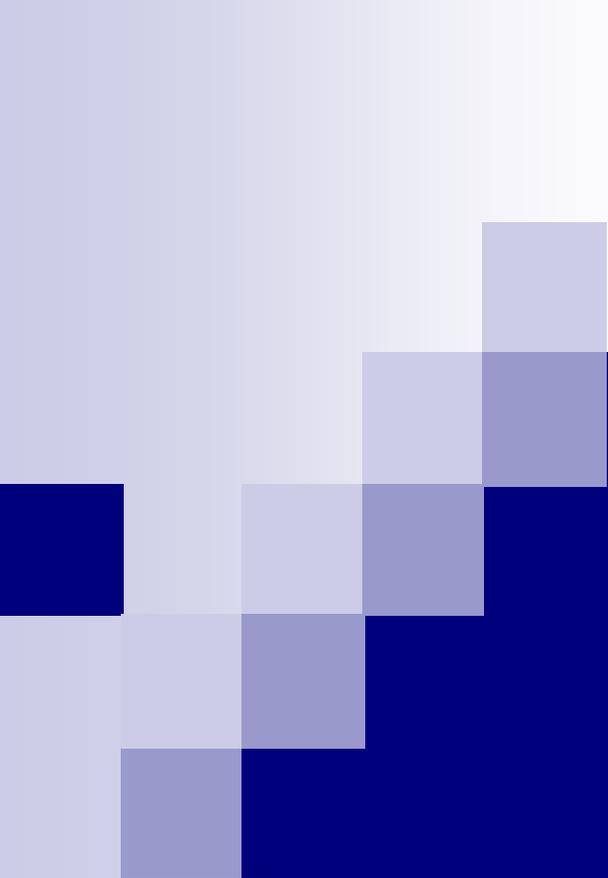
Strong Axioms of Infinity

- Dana Scott: If in \mathcal{V} there exist “very large” *large* objects (“measurable cardinals”), then \mathcal{V} is *very different* from \mathcal{L} .
- Ronald Jensen: If in \mathcal{V} there does not exist “small” *large* objects (“Not $0^\#$ ”), then \mathcal{V} and \mathcal{L} are *quite alike*.
- But Gödel’s work implies that truth of strong axioms of infinity cannot be proved.

Gödel's Belief

- There is a real mathematical universe, and it is *NOT* \mathcal{L} .
- In the real universe, the Continuum Hypothesis is *FALSE*.
- Is belief in mathematical reality ultimately reduced to a matter of faith?

Maybe not...



Reality at the Computing Level

*Is the human mind a
computer?*

Limit of the Power to Compute

- Alan Turing (1912—1954):
- *There is no algorithm to decide, for every computer program and every input, whether the program halts on that input.*

—The Halting Problem



Complex Dynamical Systems: Julia Sets J_c

$$f(z) = z^2 + c, \quad c \text{ a complex number}$$

$$f^{(2)}(z) = f(f(z)) = (z^2 + c)^2 + c$$

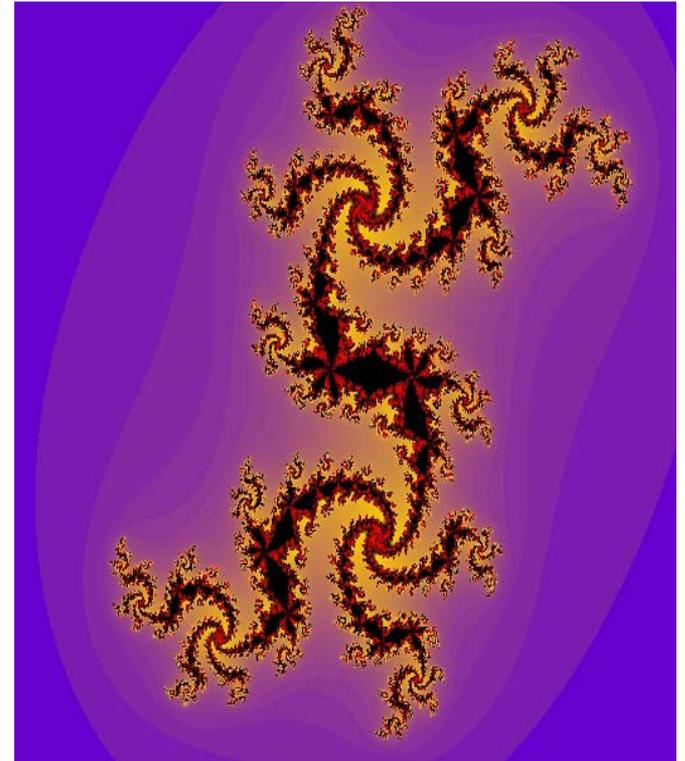
$$f^{(3)}(z) = f(f^{(2)}(z)) = (((z^2 + c)^2 + c)^2 + c$$

...

$$f^{(n)}(z) = f(f^{(n-1)}(z))$$

The Julia set of f is the boundary of $\{z \mid f^{(n)}(z) \not\rightarrow \infty\}$

Picture: $c = 0.300283 + 0.48857i$.



Computability of Complex Dynamical Systems

- A Julia set J_c is *computable* if it can be plotted on the computer screen with arbitrary precision.
- Is a J_c always computable?
- Consider the general form

$$f(z) = z^2 + cz, \quad c \text{ a complex number}$$

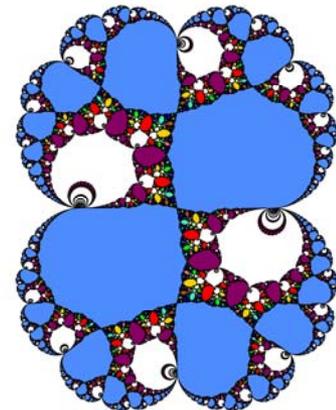
Computability of J_c

- A complex number c is computable if there is an algorithm to compute c with arbitrary precision.
- “Hyperbolic” and “Parabolic” Julia sets are (polynomial time) computable, regardless of the complexity of c .

Computability of J_c

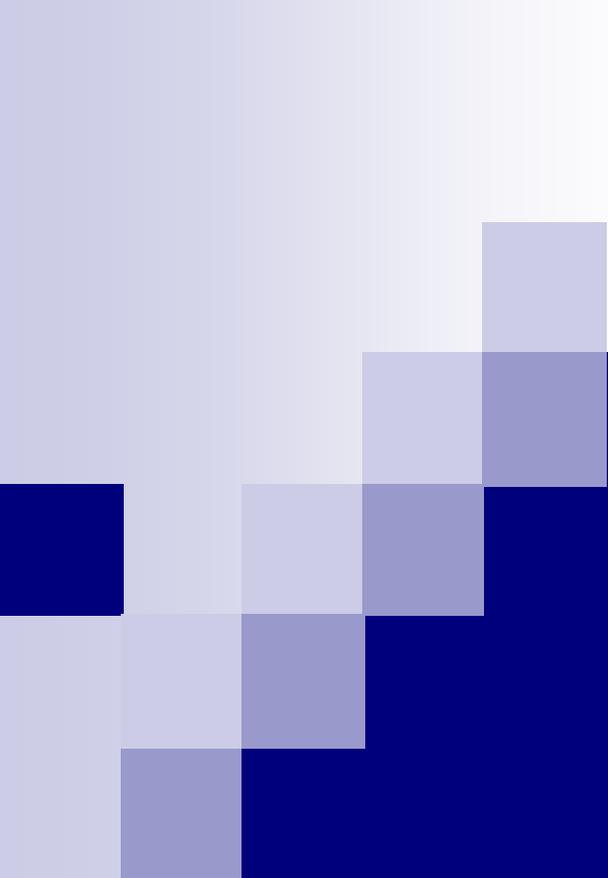
- Mark Braverman and Michael Yampolsky (*Journal of American Math Society* (2006) and *ACM Symp on Theory of Computing* (2007)):
- There is a collection of computable c 's whose J_c ("irrational Siegel disks") is *NOT* computable:

$$c = e^{2\pi i\theta}, \theta \text{ irrational}$$



Computability of J_c

- In fact, J_c can be as non-computable as the Halting Problem.
- So if Julia sets *have a reality*, and best visualized through a computer screen using numerical computations, then computer visualization is *necessarily incomplete*.



Another Level Down

Reality in the Virtual World

Science News

 Share  Blog  Cite

Making Hair Realistic In Computer Animation

ScienceDaily (Jul. 20, 2006) — Poets and novelists often describe hair as "shining" or "shimmering." Dark hair has a "sheen"; blond hair "glows." All this comes about because of the complex scattering of incident light off of individual hairs and from one hair to another.

See also:

Computers & Math

- Computer Science
- Computer Modeling
- Distributed Computing
- Computer Graphics
- Artificial Intelligence
- Mathematics

Reference

- 3D computer graphics
- Computer animation
- Computer-generated

Reproducing this effect in computer graphics has always been a challenge. Computers can create three-dimensional structures resembling hair, but the process of "rendering," in which the computer figures out how light will be reflected from those structures to create an image, requires complex calculations that take into account the scattering between hairs. Current methods use approximations that work well for dark hair and passably for brown, but computer-generated blondes still don't look like they're having more



On the left, a computer-generated image of blond hair with only direct illumination. At right, the same image rendered with the new algorithm that takes into account the multiple scattering of light through

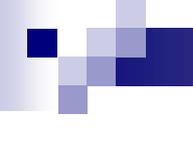
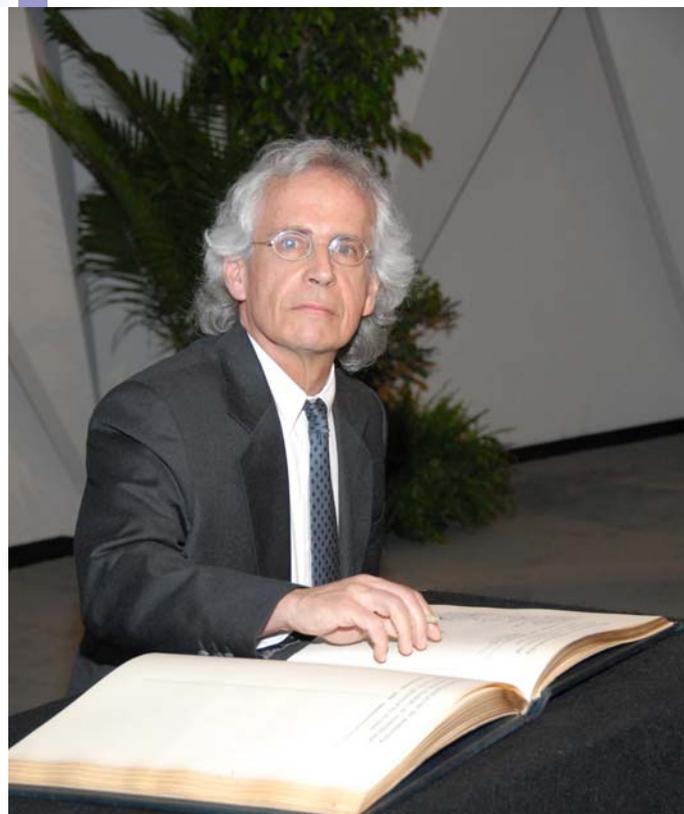


Photo-realistic Simulations

To simulate dynamic interfaces—water, fire, waves, bubbles...—require:

- Mathematically model physical phenomena (Navier-Stokes partial differential equations, optimization, probabilistic analysis, wavelets, quaternions,, ...)
- Techniques to address numerical dissipation (Level Set Method, Particle Level Set Method...)
- Fast and efficient (complex) computer algorithms for large scale (parallel) numerical calculations

These are *challenging problems*.



*Stanley Osher, UCLA
National Academy of
Sciences Award Ceremony*



James A. Sethian

2004 Wiener Prize

The 2004 AMS-SIAM Norbert Wiener Prize in Applied Mathematics was awarded at the 110th Annual Meeting of the AMS in Phoenix in January 2004.

The Wiener Prize is awarded every three years to recognize outstanding contributions to applied mathematics in the highest and broadest sense (until 2001 the prize was awarded every five years). Established in 1967 in honor of Norbert Wiener (1894-1964), the prize was endowed by the Department of Mathematics of the Massachusetts Institute of Technology. The prize is given jointly by the

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Simulation and Animation of Fire and Other Natural Phenomena in the Visual Effects Industry

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and then the velocity field at the new time step, $\mathbf{V}^{n+1} = \langle u^{n+1}, v^{n+1}, w^{n+1} \rangle$, is defined by

$$\frac{\mathbf{V}^{n+1} - \mathbf{V}^*}{\Delta t} + \frac{\nabla p}{\rho} = 0 \quad (16)$$

so that by combining equations 15 and 16 to eliminate \mathbf{V}^* results in a velocity field which satisfies equation 1. Taking the divergence of equation 16 gives

$$\nabla \cdot \left(\frac{\nabla p}{\rho} \right) = \frac{\nabla \cdot \mathbf{V}^*}{\Delta t} \quad (17)$$

after setting $\nabla \cdot \mathbf{V}^{n+1}$ to zero. Equations 16 and 17 can be rewritten as

$$\mathbf{V}^{n+1} - \mathbf{V}^* + \frac{\nabla p^*}{\rho} = 0 \quad (18)$$

and

$$\nabla \cdot \left(\frac{\nabla p^*}{\rho} \right) = \nabla \cdot \mathbf{V}^* \quad (19)$$

eliminating their dependence on Δt by using a scaled pressure, $p^* = p\Delta t$.

1 Abstract

In this article, we give simulate and animate industry. First, we disc fronts. Then, we illustr computer graphics pu dimensional Euler equ such as the nuclear ex Machines". We also l computer graphics ser explosions. Finally, we its application to the

A scientific Oscar goes to Stanford

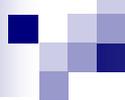
On February 9, 2008, the Academy of Motion Picture Arts and Sciences will give its ten Scientific and Technical Academy Awards for the year. One of them will go to a professor of computer science at Stanford University who worked with scientists from Industrial Light & Magic. They'll receive this 'Oscar' for their work about cyber-fluids.

They've started to work several years ago for the development of the female liquid terminator in *Terminator 3*. This technology also was used to model the sinking ship in *Poseidon* and in the two latest *Pirates of the Caribbean* movies. But read more...



Ron Fedkiw
Stanford University

You can see above several computer-generated scenes showing off the fluid simulation technology which will receive a scientific Academy Award this week at The Beverly Wilshire. (Credit: Ron Fedkiw and his team, Stanford



Two Questions

- Is there a limit to computer simulating reality? Recall simulation of Julia sets.
- Is reality different from simulation?

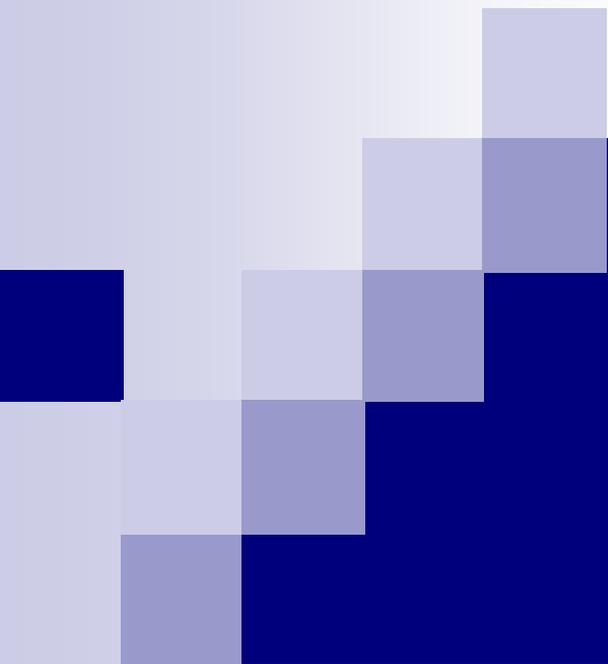
Infinity and Reality

- If reality is finite, why and how is infinity conceived by the mind?
- Why is the infinite in mathematics so useful in the study and understanding of finite reality?
- If reality is effectively infinite, and inspires as well as understood through mathematics, is *it the manifestation of this reality?*

- *Wir müssen wissen*
- *Wir werden wissen*

David Hilbert (1900)





What Is Reality?